*Electronic Journal of Qualitative Theory of Differential Equations*
2012, No. **41**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

# On existence of positive solutions of coupled integral boundary value problems for a nonlinear singular superlinear differential system☆

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**Abstract**

By constructing a special cone and using fixed point index theory, this paper investigates the existence of positive solutions of singular superlinear coupled integral boundary value problems for differential systems

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t)), \quad t \in (0, 1), \\ -y''(t) = f_2(t, x(t), y(t)), \quad t \in (0, 1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x], \end{cases}$$

where $\alpha[x], \beta[x]$ are bounded linear functionals on $C[0, 1]$ given by

$$\alpha[x] = \int_0^1 x(t)dA(t), \quad \beta[x] = \int_0^1 x(t)dB(t)$$

with $A, B$ functions of bounded variation with positive measures.

*Key words:* Positive solutions, Coupled singular system, Coupled integral boundary conditions, Fixed point index theory

**MSC** 34B10, 34B18

☆The Project Supported by the National Science Foundation of P.R.China(10971179, 11126094) and Research Award Fund for Outstanding Young Scientists of Shandong Province(BS2010SF023)

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## 1. Introduction

In this paper, we consider the following nonlinear singular second order ordinary differential system (ODS for short) with coupled integral boundary value conditions

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t)), & t \in (0,1), \\ -y''(t) = f_2(t, x(t), y(t)), & t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x], \end{cases} \tag{1.1}$$

where $f_1$ and $f_2 : (0,1) \times [0, +\infty)^2 \to [0, +\infty)$ are continuous and may be singular at $t = 0, 1$; $\alpha[x], \beta[x]$ are bounded linear functionals on $C[0,1]$ given by

$$\alpha[x] = \int_0^1 x(t)dA(t), \quad \beta[x] = \int_0^1 x(t)dB(t),$$

involving Stieltjes integrals, in particular, $A, B$ are functions of bounded variation with positive measures.

Boundary value problems for an ODS arise from many fields in physics, biology and chemistry, which play a very important role in both theory and application. In recent years, there were many works to be done for a variety of nonlinear second order ordinary differential systems. However, most papers only focus on attention to the differential system with uncoupled boundary conditions; we refer the readers to [1, 2, 3, 5, 8, 9, 10, 11, 12, 13, 14, 15, 17, 20, 21, 25] and the reference therein. On the other hand, there are several model problems where the differential system are coupled not only in the differential system but also through the boundary conditions ([24, 27]). In a recent article [4] the author studied the following singular system with coupled four-point boundary value conditions

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t)), & t \in (0,1), \\ -y''(t) = f_2(t, x(t), y(t)), & t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha y(\xi), \quad y(1) = \beta x(\eta). \end{cases}$$

By using the Guo-Krasnosel'skiĭ fixed-point theorem [7], some existence results were obtained when the nonlinearities $f_1$ and $f_2$ are sublinear in $x$ and $y$. In

[26], the authors considered the existence of positive solutions of systems of the nonlinear semipositone fractional differential equation with four-point coupled boundary value problem

$$\begin{cases} \mathrm{D}_{0+}^{\alpha}u + \lambda f(t,u,v) = 0, \quad t \in (0,1), \lambda > 0, \\ \mathrm{D}_{0+}^{\alpha}v + \lambda g(t,u,v) = 0, \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ u(1) = av(\xi), \quad v(1) = bu(\eta), \end{cases}$$

where $\lambda$ is a parameter, $a, b, \xi, \eta$ satisfy $\xi, \eta \in (0,1)$, $0 < ab\xi\eta < 1$, $\alpha \in (n-1, n]$ is a real number and $n \geq 3$, and $\mathrm{D}_{0+}^{\alpha}u$ is the Riemann-Liouville's fractional derivative. They established the existence results by a nonlinear alternative of Leray-Schauder type and Guo-Krasnosel'skiĭ fixed-point theorem in a cone.

Nonlocal boundary value problems have been well studied especially on a compact interval. For example, Webb and Infante have made an extensive study of nonlocal boundary value problems involving integral conditions in [18, 19] by giving a general approach to cover many nonlocal boundary conditions in a unified way. We should note that the work of Webb and Infante does not require the functionals $\alpha[x], \beta[x]$ to be positive for all positive $x$.

To the best of our knowledge, differential system (1.1) has not been treated in the superlinear case even for uncoupled boundary conditions. Motivated by [4, 18, 19], the purpose of this paper is to establish the existence of at least one positive solution for differential system with coupled integral boundary value problems (1.1) when the nonlinearities $f_1$ and $f_2$ are superlinear in $x$ and $y$. By a positive solution of the system (1.1), we mean that $(x, y) \in (C[0,1] \cap C^2(0,1)) \times (C[0,1] \cap C^2(0,1))$, $(x, y)$ satisfies (1.1), $x > 0$ and $y > 0$ on $(0,1]$.

Throughout the paper, we assume that the following conditions hold:

$(H_1)$ $f_i \in C((0,1) \times [0,\infty)^2, [0,\infty))(i = 1,2)$ and satisfy

$$0 < \int_0^1 s(1-s) f_1(s,1,1)ds < +\infty, \; 0 < \int_0^1 s(1-s) f_2(s,1,1)ds < +\infty.$$

$(H_2)$ There exist constants $\lambda_{ij}, \mu_{ij} (0 < \lambda_{ij} \leq \mu_{ij}, i, j = 1, 2,\ \Sigma_{j=1}^{2} \lambda_{ij} > 1, i =$

1, 2) such that for $t \in (0,1)$, $x, y \in (0,\infty)$,

$$c^{\mu_{i1}} f_i(t,x,y) \le f_i(t,cx,y) \le c^{\lambda_{i1}} f_i(t,x,y), \quad \text{if } 0 < c \le 1,\ i = 1,2, \tag{1.2}$$

$$c^{\mu_{i2}} f_i(t,x,y) \le f_i(t,x,cy) \le c^{\lambda_{i2}} f_i(t,x,y), \quad \text{if } 0 < c \le 1,\ i = 1,2. \tag{1.3}$$

$(H_3)$ $\alpha[t] = \displaystyle\int_0^1 t dA(t) > 0,\ \ \beta[t] = \int_0^1 t dB(t) > 0,\ \ \kappa = 1 - \alpha[t]\beta[t] > 0.$

**Remark 1.1**. Condition $(H_2)$ is used to discuss the existence of positive solutions of higher-order differential equations/system. We refer the reader to [21, 22, 23] for sublinear case ($\Sigma_{j=1}^2 \mu_{ij} < 1, i = 1,2$) and to [6, 16] for superlinear case ($\Sigma_{j=1}^2 \lambda_{ij} > 1, i = 1,2$).

(i) (1.2) and (1.3) implies

$$c^{\lambda_{i1}} f_i(t,x,y) \le f_i(t,cx,y) \le c^{\mu_{i1}} f_i(t,x,y), \quad \text{if } c \ge 1,\ i = 1,2, \tag{1.4}$$

$$c^{\lambda_{i2}} f_i(t,x,y) \le f_i(t,x,cy) \le c^{\mu_{i2}} f_i(t,x,y), \quad \text{if } c \ge 1,\ i = 1,2. \tag{1.5}$$

Conversely, (1.4) implies (1.2), (1.5) implies (1.3).

(ii) (1.2) and (1.3) implies

$$f_i(t,x_1,x_2) \le f_i(t,y_1,y_2), \ \text{ if } \ 0 < x_j \le y_j, \ \ i,j = 1,2. \tag{1.6}$$

**Remark 1.2**. Typical functions that satisfy the above superlinear hypothesis are those taking the form $f_i(t,x,y) = \Sigma_{j=1}^m p_{ij}(t) x^{\lambda_{i1j}} y^{\lambda_{i2j}}$; here $p_{ij}(t) \in C(0,1)$, $p_{ij}(t) > 0$ on $(0,1)$, $\lambda_{i1j} > 0$, $\lambda_{i2j} > 0$, $\lambda_{i1j} + \lambda_{i2j} > 1$, $i = 1,2, j = 1,2,\ldots,m$.

The rest of paper is organized as follows. In section 2, we shall give some preliminary results and lemmas to prove our main results. In section 3, we establish the existence results of at least one positive solution for differential system (1.1) by fixed point index theory on cones.

## 2. Preliminaries

For each $u \in E := C[0,1]$, we write $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. Clearly, $(E, \|\cdot\|)$ is a Banach space. For each $(x,y) \in E \times E$, we write $\|(x,y)\|_1 = \max\{\|x\|, \|y\|\}$. Define

$$P = \{(x,y) \in E \times E : x(t) \ge \gamma t \|(x,y)\|_1, y(t) \ge \gamma t \|(x,y)\|_1, \ \ t \in [0,1].\},$$

where

$$0<\gamma=\frac{\nu}{\rho}<1,\quad \rho=\max\left\{\frac{\alpha[t]}{\kappa}\beta[1]+1,\frac{\beta[t]}{\kappa}\alpha[1]+1,\frac{1}{\kappa}\beta[1],\frac{1}{\kappa}\alpha[1]\right\},$$

$$\nu=\min\left\{\frac{\alpha[t]}{\kappa}\beta[t(1-t)],\frac{\beta[t]}{\kappa}\alpha[t(1-t)],\ \frac{1}{\kappa}\beta[t(1-t)],\ \frac{1}{\kappa}\alpha[t(1-t)]\right\}.$$

Clearly, $(E\times E,\|\cdot\|_1)$ is a Banach space and $P$ is a cone of $E\times E$. For any real constant $r>0$, define $\Omega_r=\{(x,y)\in P:\|(x,y)\|_1<r\}$.

**Lemma 2.1**. *Let $u,v\in E$, then the differential system of BVPs*

$$\begin{cases} -x''(t)=u(t),\quad -y''(t)=v(t),\quad t\in[0,1],\\ x(0)=y(0)=0,\quad x(1)=\alpha[y],\quad y(1)=\beta[x] \end{cases}\tag{2.1}$$

*has integral representation*

$$\begin{cases} x(t)=\int_0^1 G_1(t,s)u(s)ds+\int_0^1 H_1(t,s)v(s)ds,\\ y(t)=\int_0^1 G_2(t,s)v(s)ds+\int_0^1 H_2(t,s)u(s)ds, \end{cases}\tag{2.2}$$

*where*

$$G_1(t,s)=\frac{\alpha[t]t}{\kappa}\int_0^1 K(s,\tau)dB(\tau)+K(t,s),\ H_1(t,s)=\frac{t}{\kappa}\int_0^1 K(s,\tau)dA(\tau),$$

$$G_2(t,s)=\frac{\beta[t]t}{\kappa}\int_0^1 K(s,\tau)dA(\tau)+K(t,s),\ H_2(t,s)=\frac{t}{\kappa}\int_0^1 K(s,\tau)dB(\tau),$$

$$K(t,s)=\begin{cases} t(1-s),\quad 0\le t\le s\le 1,\\ s(1-t),\quad 0\le s\le t\le 1. \end{cases}$$

**Proof.** It is easy to see that (2.1) is equivalent to the system of integral equations

$$x(t)=x(1)t+\int_0^1 K(t,s)u(s)ds,\quad t\in[0,1],\tag{2.3}$$

$$y(t)=y(1)t+\int_0^1 K(t,s)v(s)ds,\quad t\in[0,1].\tag{2.4}$$

Applying $\beta$ and $\alpha$ to (2.3) and (2.4) respectively we obtain

$$\int_0^1 x(t)dB(t)=x(1)\int_0^1 tdB(t)+\int_0^1\int_0^1 K(t,s)u(s)dsdB(t),$$

$$\int_0^1 y(t)dA(t) = y(1)\int_0^1 tdA(t) + \int_0^1\int_0^1 K(t,s)v(s)dsdA(t).$$

Therefore,

$$\begin{pmatrix} -\beta[t] & 1 \\ 1 & -\alpha[t] \end{pmatrix}\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} \int_0^1\int_0^1 K(t,s)u(s)dsdB(t) \\ \int_0^1\int_0^1 K(t,s)v(s)dsdA(t) \end{pmatrix}$$

and so

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \frac{1}{\kappa}\begin{pmatrix} \alpha[t] & 1 \\ 1 & \beta[t] \end{pmatrix}\begin{pmatrix} \int_0^1\int_0^1 K(t,s)u(s)dsdB(t) \\ \int_0^1\int_0^1 K(t,s)v(s)dsdA(t) \end{pmatrix}. \tag{2.5}$$

Substituting (2.5) into (2.3) and (2.4), we have

$$\begin{aligned} x(t) &= \frac{\alpha[t]t}{\kappa}\int_0^1\int_0^1 K(t,s)u(s)dsdB(t) + \frac{t}{\kappa}\int_0^1\int_0^1 K(t,s)v(s)dsdA(t) \\ &\quad + \int_0^1 K(t,s)u(s)ds, \\ y(t) &= \frac{t}{\kappa}\int_0^1\int_0^1 K(t,s)u(s)dsdB(t) + \frac{\beta[t]t}{\kappa}\int_0^1\int_0^1 K(t,s)v(s)dsdA(t) \\ &\quad + \int_0^1 K(t,s)v(s)ds, \end{aligned}$$

which is equivalent to the system (2.2).□

**Remark 2.1**. It is easy to show that the function $K(t,s)$ has the following properties:

$$t(1-t)s(1-s) \le K(t,s) = K(s,t) \le s(1-s), \quad \forall\ t,s \in [0,1].$$

From this and $(H_3)$, for $t \in [0,1]$, we have

$$G_i(t,s) \le \rho s(1-s),\ H_i(t,s) \le \rho s(1-s),\ i = 1,2, \tag{2.6}$$

and

$$G_i(t,s) \ge \nu ts(1-s),\ H_i(t,s) \ge \nu ts(1-s),\ i = 1,2. \tag{2.7}$$

Define an operator $T : P \to Q \times Q$ by

$$T(x,y) = (T_1(x,y), T_2(x,y)),$$

where operators $T_1, T_2 : P \to Q = \{u \in E \mid u(t) \geq 0,\ t \in [0,1]\}$ are defined by

$$T_1(x,y)(t) = \int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 H_1(t,s)f_2(s,x(s),y(s))ds,\ t \in [0,1],$$

$$T_2(x,y)(t) = \int_0^1 G_2(t,s)f_2(s,x(s),y(s))ds + \int_0^1 H_2(t,s)f_1(s,x(s),y(s))ds,\ t \in [0,1].$$

For $(x,y) \in P$, let $c$ be a positive number such that $\frac{\|(x,y)\|_1}{c} < 1$ and $c > 1$. From (1.5) and (1.6), we have

$$f_i(t,x(t),y(t)) \leq f_i(t,c,c) \leq c^{\mu_{i1}+\mu_{i2}} f_i(t,1,1), \quad i = 1,2.$$

Hence for any $t \in [0,1]$, by Remark 2.1, we get

$$\begin{aligned} T_i(x,y)(t) \quad &\leq \rho \int_0^1 s(1-s)f_1(s,x(s),y(s))ds + \rho \int_0^1 s(1-s)f_2(s,x(s),y(s))ds \\ &\leq \rho c^{\mu_{11}+\mu_{12}} \int_0^1 s(1-s)f_1(s,1,1)ds + \rho c^{\mu_{21}+\mu_{22}} \int_0^1 s(1-s)f_2(s,1,1)ds,\ i = 1,2. \end{aligned}$$

Thus, if $(H_1)$ and $(H_2)$ hold, $T$ is well defined on $P$. Moreover, by Lemma 2.1, if $(x,y) \in P$ is a fixed point of $T$, then $(x,y)$ is a solution of differential system (1.1).

**Lemma 2.2**.*If $(H_1)$ and $(H_2)$ hold, then $T(P) \subset P$.*

**Proof.** By Remark 2.1, for $\tau, t, s \in [0,1]$, we obtain

$$G_i(t,s) \geq \gamma t G_i(\tau,s), \quad H_i(t,s) \geq \gamma t H_i(\tau,s),\ i = 1,2,$$

$$H_1(t,s) \geq \gamma t G_2(\tau,s), \quad G_1(t,s) \geq \gamma t H_2(\tau,s)$$

and

$$H_2(t,s) \geq \gamma t G_1(\tau,s), \quad G_2(t,s) \geq \gamma t H_1(\tau,s).$$

Hence, for $(x,y) \in P$, $t \in [0,1]$, we have

$$\begin{aligned} T_1(x,y)(t) \quad &= \int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 H_1(t,s)f_2(s,x(s),y(s))ds \\ &\geq \gamma t \int_0^1 G_1(\tau,s)f_1(s,x(s),y(s))ds + \gamma t \int_0^1 H_1(\tau,s)f_2(s,x(s),y(s))ds \\ &= \gamma t T_1(x,y)(\tau) \end{aligned}$$

and

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 G_1(t,s)f_1(s,x(s),y(s))ds + \int_0^1 H_1(t,s)f_2(s,x(s),y(s))ds \\ &\geq \gamma t \int_0^1 H_2(\tau,s)f_1(s,x(s),y(s))ds + \gamma t \int_0^1 G_2(\tau,s)f_2(s,x(s),y(s))ds \\ &= \gamma t T_2(x,y)(\tau). \end{aligned}$$

Then $T_1(x,y)(t) \geq \gamma t \|T_1(x,y)\|$ and $T_1(x,y)(t) \geq \gamma t \|T_2(x,y)\|$, i.e.,. $T_1(x,y)(t) \geq \gamma t \|(T_1(x,y), T_2(x,y))\|_1$. In the same way, we can prove that $T_2(x,y)(t) \geq \gamma t \|(T_1(x,y), T_2(x,y))\|_1$. Therefore, $T(P) \subset P$.□

**Lemma 2.3** *If $(H_1)$ and $(H_2)$ hold, then $T$ is a completely continuous operator on $P$.*

**Proof.** This is a standard textbook result using Ascoli-Arzela theorem, see for example [7], and is omitted.

## 3. Main result

**Theorem 3.1**. *If $(H_1)-(H_3)$ hold, the differential system (1.1) has at least one positive solution.*

**Proof.** Choose a constant $R > 0$ such that

$$R > \max\{\frac{1}{\gamma}+1, \left(\sigma\gamma^{\lambda_{11}+\lambda_{12}}\right)^{-\frac{1}{\lambda_{11}+\lambda_{12}-1}}, \left(\sigma\gamma^{\lambda_{21}+\lambda_{22}}\right)^{-\frac{1}{\lambda_{21}+\lambda_{22}-1}}\},$$

where $\sigma = \dfrac{\nu}{4}\displaystyle\int_0^1 (\gamma s)^{\mu_{11}+\mu_{12}} s(1-s) f_1(s,1,1)ds > 0$.

We may suppose that $T$ has no fixed point on $\partial\Omega_R$ (otherwise, the proof is finished). Now we show that

$$(x,y) - T(x,y) \neq \tau(t,t), \quad \forall\ (x,y) \in \partial\Omega_R,\ \tau \geq 0. \tag{3.1}$$

If otherwise, there exist $(x_1,y_1) \in \partial\Omega_R$ and $\tau_1 > 0$ such that

$$(x_1,y_1) - T(x_1,y_1) = \tau_1(t,t),$$

that is,

$$x_1 = T_1(x_1,y_1) + \tau_1 t, \quad y_1 = T_2(x_1,y_1) + \tau_1 t. \tag{3.2}$$

Without loss of generality, we assume that $\|x_1\| = \|(x_1, y_1)\|_1 > \frac{1}{\gamma}$. By the definition of $P$, we have $\|y_1\| > 1$.

Let $E_1 = \{s \in [0,1] : s\gamma\|x_1\| \leq 1\}$, $E_2 = \{s \in [0,1] : s\gamma\|x_1\| > 1\}$, $F_1 = \{s \in [0,1] : s\gamma\|y_1\| \leq 1\}$, $F_2 = \{s \in [0,1] : s\gamma\|y_1\| > 1\}$. Clearly, $E_1 \subset F_1$, $E_1 \neq \emptyset$, $E_2 \neq \emptyset$ and $F_2 \neq \emptyset$.

By (3.2), $(H_2)$ and (2.7), for $t \in [\frac{1}{4}, 1]$, we obtain

$$\begin{aligned}
x_1(t) &\geq T_1(x_1, y_1)(t) \geq \int_0^1 G_1(t,s) f_1(s, x_1(s), y_1(s))ds \\
&\geq \frac{\nu}{4} \int_0^1 s(1-s) f_1(s, \gamma\|x_1\|s, \gamma\|y_1\|s)ds \\
&= \frac{\nu}{4}\Big(\int_{E_1\cap F_1} + \int_{E_2\cap F_1} + \int_{E_2\cap F_2}\Big) \\
&\geq \frac{\nu}{4}\Big(\int_{E_1\cap F_1} s(1-s)(s\gamma\|x_1\|)^{\mu_{11}}(s\gamma\|y_1\|)^{\mu_{12}} f_1(s,1,1)ds \\
&\quad + \int_{E_2\cap F_1} s(1-s)(s\gamma\|x_1\|)^{\lambda_{11}}(s\gamma\|y_1\|)^{\mu_{12}} f_1(s,1,1)ds \\
&\quad + \int_{E_2\cap F_2} s(1-s)(s\gamma\|x_1\|)^{\lambda_{11}}(s\gamma\|y_1\|)^{\lambda_{12}} f_1(s,1,1)ds\Big) \\
&\geq \frac{\nu}{4}\Big(\int_{E_1\cap F_1} s(1-s)(\gamma s)^{\mu_{11}+\mu_{12}}\|x_1\|^{\lambda_{11}}\|y_1\|^{\lambda_{12}} f_1(s,1,1)ds \\
&\quad + \int_{E_2\cap F_1} s(1-s)(\gamma s)^{\mu_{11}+\mu_{12}}\|x_1\|^{\lambda_{11}}\|y_1\|^{\lambda_{12}} f_1(s,1,1)ds \\
&\quad + \int_{E_2\cap F_2} s(1-s)(\gamma s)^{\mu_{11}+\mu_{12}}\|x_1\|^{\lambda_{11}}\|y_1\|^{\lambda_{12}} f_1(s,1,1)ds\Big) \\
&\geq \frac{\nu}{4}\|x_1\|^{\lambda_{11}}\|y_1\|^{\lambda_{12}} \int_0^1 s(1-s)(\gamma s)^{\mu_{11}+\mu_{12}} f_1(s,1,1)ds \\
&= \sigma\|x_1\|^{\lambda_{11}}\|y_1\|^{\lambda_{12}}.
\end{aligned}$$

Consequently,

$$\|(x_1, y_1)\|_1 = \|x_1\| \geq \sigma\|x_1\|^{\lambda_{11}}\|y_1\|^{\lambda_{12}} \geq \sigma\gamma^{\lambda_{11}+\lambda_{12}}\|(x_1, y_1)\|_1^{\lambda_{11}+\lambda_{12}}. \tag{3.3}$$

namely

$$R = \|(x_1, y_1)\|_1 \leq \left(\sigma\gamma^{\lambda_{11}+\lambda_{12}}\right)^{-\frac{1}{\lambda_{11}+\lambda_{12}-1}}$$

which is a contradiction.

Summing up, (3.1) is true and by properties of fixed point index we have

$$i(T, \Omega_R, P) = 0. \tag{3.4}$$

Next we claim that

$$T(x, y) \neq \tau(x, y), \quad \forall\ (x, y) \in \partial\Omega_r, \quad \tau \geq 1, \tag{3.5}$$

where

$$0 < r < \min\{\frac{1}{2}, \delta^{-\frac{1}{\lambda-1}}\}, \quad \lambda = \min\{\lambda_{11} + \lambda_{12}, \lambda_{21} + \lambda_{22}\} > 1,$$

$$\delta = \rho\left(\int_0^1 s(1-s)f_1(s, 1, 1)ds + \int_0^1 s(1-s)f_2(s, 1, 1)ds\right).$$

If otherwise, there exist $(x_2, y_2) \in \partial\Omega_r$ and $\tau_2 \geq 1$ such that

$$T(x_2, y_2) = \tau_2(x_2, y_2). \tag{3.6}$$

We may suppose that $\tau_2 > 1$, otherwise $T$ has a fixed point on $\partial\Omega_r$ and the proof is finished. Without loss of generality, we assume that $\|x_2\| = \|(x_2, y_2)\|_1 = \max\{\|x_2\|, \|y_2\|\} = r$. By the definition of $P$, we have $0 < \gamma r \leq \|y_2\| \leq r < 1$. From (3.6) and (2.6), it follows that

$$\begin{aligned}
&\tau_2 x_2(t) = T_1(x_2, y_2)(t) \\
= &\int_0^1 G_1(t, s)f_1(s, x_2(s), y_2(s))ds + \int_0^1 H_1(t, s)f_2(s, x_2(s), y_2(s))ds \\
\leq &\ \rho\int_0^1 s(1-s)f_1(s, r, r)ds + \rho\int_0^1 s(1-s)f_2(s, r, r)ds \\
\leq &\ \rho r^{\lambda_{11}+\lambda_{12}}\int_0^1 s(1-s)f_1(s, 1, 1)ds + \rho r^{\lambda_{21}+\lambda_{22}}\int_0^1 s(1-s)f_2(s, 1, 1)ds \\
\leq &\ \delta r^{\lambda}, \quad t \in [0, 1].
\end{aligned}$$

Consequently,

$$r = \|x_2\| < \tau_2\|x_2\| \leq \delta r^{\lambda},$$

namely

$$r \geq \delta^{-\frac{1}{\lambda-1}},$$

which is a contradiction. Hence (3.5) is true and by properties of fixed point index we have

$$i(T, \Omega_r, P) = 1. \tag{3.7}$$

By (3.4) and (3.7) we have

$$i(T, \Omega_R \backslash \overline{\Omega_r}, P) = i(T, \Omega_R, P) - i(T, \Omega_r, P) = -1.$$

Then $T$ has at least one fixed on $\Omega_R \backslash \overline{\Omega_r}$. This means that differential system (1.1) has at least one positive solution.

**Acknowledgments**

The authors sincerely thank the reviewer for careful reading and useful comments that have led to the present improved version of the original paper.

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(Received December 22, 2011)